Exercise 8.5.3

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02407 Stochastic Processes

This exercise presents an excellent opportunity to see how the theory of stochastic processes are applicable in other courses. In this specific instance, how the theory relates to the content of the course 02417 Time Series Analysis, in which they cover the subject of autoregressive (AR) processes.

Autoregressive models are relevant here because this exercise essentially shows that the Ornstein-Uhlenbeck process can be considered as a continuous version of an AR(1)-model.

The AR(1)-model $\{V_n\}_{n\in\mathbb{N}_0}$ is defined through the recursion:

$$V_n = (1-\beta)V_{n-1} + \xi_n, \quad V_0 = \nu,$$

where $0 < \beta < 1$ and the random variables ξ_i $(i = 0, 1, ...)$ are independent and identically distributed with a standard normal distribution.

We start by determining the mean value function. We achieve this by establishing a recursion for the mean value:

$$\mathbb{E}[V_n] = \mathbb{E}\left[(1-\beta)V_{n-1} + \xi_n\right] = (1-\beta)\mathbb{E}[V_{n-1}] + \mathbb{E}[\xi_n] = (1-\beta)\mathbb{E}[V_{n-1}]. \tag{1}$$

In the above, we used the linearity of the expectation operator and the fact that $\mathbb{E}[\xi_n] = 0$. Note that the above relationship holds for an arbitrary $n \in \mathbb{N}$. Consequently, $\mathbb{E}[V_n] = 0$ iff. $\mathbb{E}[V_{n-1}] = 0$, and by applying this principle successively, $\mathbb{E}[V_n] = 0$ iff. $\nu = 0$. Therefore, if $\nu = 0$ we have that $\mathbb{E}[V_n] = 0$ for all $n \in \mathbb{N}$.

If we assume the converse, i.e. $\nu \neq 0$, then we know that $\mathbb{E}[V_n] \neq 0$ for all $n \in \mathbb{N}$. In that case, we can rewrite eq. (1) to

$$\frac{\mathbb{E}[V_n]}{\mathbb{E}[V_{n-1}]} = 1 - \beta, \quad \forall n \in \mathbb{N}.$$

It follows that

$$\frac{\mathbb{E}[V_n]}{\mathbb{E}[V_0]} = \prod_{i=1}^{n} \frac{\mathbb{E}[V_i]}{\mathbb{E}[V_{i-1}]} = \prod_{i=1}^{n} (1-\beta) = (1-\beta)^n,$$

leading to the result that $\mathbb{E}[V_n] = (1-\beta)^n \mathbb{E}[V_0] = (1-\beta)^n \nu$. We see that this result also holds for $\nu = 0$ and we conclude that the result holds in all generality. If you have solved this exercise using proofs by induction that is perfectly acceptable too.

We next calculate the (auto)covariance function for AR(1)-process. First we note that

$$\mathrm{Cov}(V_n, V_{n+k}) = \mathrm{Cov}(V_n, (1-\beta)V_{n+k-1} + \xi_{n+k}) = (1-\beta)\mathrm{Cov}(V_n, V_{n+k-1}) + \mathrm{Cov}(V_n, \xi_{n+k})$$

for $n, k \in \mathbb{N}$, as the covariance is bilinear. As V_n is independent of ξ_{n+k}, their covariance is zero, which simplifies the above expression to

$$\mathrm{Cov}(V_n, V_{n+k}) = (1-\beta)\mathrm{Cov}(V_n, V_{n+k-1}).$$

If we apply this result successively, we obtain that

$$\mathrm{Cov}(V_n, V_{n+k}) = (1-\beta)^k \mathrm{Cov}(V_n, V_n) = (1-\beta)^k \mathbb{V}[V_n].$$

It remains to determine the variance of V_n. Similarly to the previous, we can formulate a (here inhomogeneous) recurrence equation

$$\mathbb{V}[V_n] = \mathbb{V}[(1-\beta)V_{n-1} + \xi_n] = (1-\beta)^2 \mathbb{V}[V_{n-1}] + 1, \quad \mathbb{V}[V_0] = 0,$$

as ξ_n has unit variance and is independent of V_{n-1}. This reccurence equation has the solution

$$\mathbb{V}[V_n] = \frac{1-(1-\beta)^{2n}}{1-(1-\beta)^2}, \tag{2}$$

which means that the covariance function takes the form:

$$\mathrm{Cov}(V_n, V_{n+k}) = (1-\beta)^k \left(\frac{1-(1-\beta)^{2n}}{1-(1-\beta)^2} \right) = \frac{(1-\beta)^k - (1-\beta)^{n+(n+k)}}{1-(1-\beta)^2}.$$

In full generality, we get that for two integers $0 < n < m$,

$$\mathrm{Cov}(V_n, V_m) = \frac{(1-\beta)^{m-n} - (1-\beta)^{n+m}}{1-(1-\beta)^2}.$$

You should not worry, if it is not immediately clear to you why eq. (2) holds. The underlying idea is to write V_n as

$$V_n = \sum_{i=1}^{n} \xi_i (1-\beta)^{n-i} + \nu(1-\beta)^n$$

by successively applying the recurrence for V_n, V_{n-1}, V_{n-2} and so on. Taking the variance of this yields

$$\begin{aligned}
\mathbb{V}[V_n] &= \mathbb{V}\left[\sum_{i=1}^{n} \xi_i (1-\beta)^{n-i} + \nu(1-\beta)^n \right] \\
&= \mathbb{V}\left[\sum_{i=1}^{n} \xi_i (1-\beta)^{n-i} \right] \\
&= \sum_{i=1}^{n} (1-\beta)^{2(n-i)} \mathbb{V}\left[\xi_i\right] \\
&= \sum_{i=1}^{n} (1-\beta)^{2(n-i)} \\
&= \sum_{j=0}^{n-1} (1-\beta)^{2j},
\end{aligned}$$

where $j = n - i$. In the first step, we use that the variance of a constant is zero, while we in step four invoke that ξ_i has unit variance. We recognice this as the first n terms of a geometric series, and from basic calculus we know the closed form solution to this is given as

$$\sum_{j=0}^{n-1}(1-\beta)^{2j} = \sum_{j=0}^{n-1}\left[(1-\beta)^2\right]^j = \frac{1-\left[(1-\beta)^2\right]^n}{1-(1-\beta)^2} = \frac{1-(1-\beta)^{2n}}{1-(1-\beta)^2}.$$

We now move on to part b of the exercise and calculate the conditional mean and variance of the process increments. We define $\Delta V_{n+1} = V_{n+1} - V_n$ and note that

$$\Delta V_{n+1} = V_{n+1} - V_n = (1-\beta)V_n + \xi_{n+1} - V_n = -\beta V_n + \xi_{n+1},$$

which allows us to determine the conditional mean of ΔV_{n+1} given $V_n = \nu$ as

$$\mathbb{E}[\Delta V_{n+1}|V_n = \nu] = \mathbb{E}[-\beta V_n + \xi_{n+1}|V_n = \nu] = -\beta\nu.$$

Similarly, we get the conditional variance as

$$\begin{aligned}\mathbb{V}[\Delta V_{n+1}|V_n = \nu] &= \mathbb{V}[-\beta V_n + \xi_{n+1}|V_n = \nu] \\ &= \mathbb{V}[-\beta\nu + \xi_{n+1}|V_n = \nu] = \mathbb{V}[\xi_{n+1}|V_n = \nu] = \mathbb{V}[\xi_{n+1}] = 1.\end{aligned}$$

We have used that constants have zero variance and the independence between ξ_{n+1} and V_n.